2010 WORDS+ / ISAAC OUTSTANDING CONSUMER LECTURE

Philosophy of mathematics – What the heck is that?

By Thor Sandmel

First, I want to thank the Outstanding Consumer Lecture Committee for granting me this award and the opportunity to speak to you today. I hope you will enjoy the lecture.

Let me start by giving a preliminary answer to the question in the title of this lecture. First: philosophy of mathematics is philosophy, and definitely not mathematics. Thus, it is not "philosophical mathematics". Nor is it "mathematical philosophy". At least in my opinion, mathematics simply cannot be philosophical, and philosophy cannot be mathematical.

So, philosophy of mathematics is neither philosophical mathematics nor mathematical philosophy. What is it, then? It is, quite simply, philosophy **about** mathematics, a branch of philosophy having mathematics as its subject matter. You wonder how philosophy can be about mathematics? Well, philosophy can be about absolutely anything at all. There is no field of human knowledge, activity or experience that does not raise "meta-questions", as to the validity, origin, genesis, and general nature of the knowledge, activity, or experience, and such meta-questions are generally called philosophical.

But enough talking, for now. I intend to prove to you that you are indeed all philosophers of mathematics. That can only be done by letting you see yourself acting as philosophers of mathematics. So, let's get some action!

I shall start by posing a few simple questions and exercises, and give you a couple of minutes to mull over them. Mind you, this is not an exam, I will not collect your answers and cover them with red marks. And there is no such thing as cheating. In fact, discussing with your neighbours is strongly encouraged, as it will almost certainly enhance your learning experience.

Here goes:

1. What mood does listening to "Summertime" from "Porgy and Bess" get you into? (If you don't know this song, feel free to choose any other song well known to you.)

2. How many people are there in this room?

3. What is worse of toothache and headache?

4. How many words are there in this sentence?

5. Is there a number which multiplied by itself equals 2?

6. Construct a plane biangle, i.e. a closed figure limited by two straight lines (the same way as a triangle is a closed figure limited by three straight lines).

7. Given an equilateral triangle, i.e. a triangle where all three edges are of equal length. Divide it into four exactly equal equilateral triangles.

Now, I'll give you a couple of minutes to think about these seven questions or exercises. Don't take this too seriously, don't fret about finding the "correct" answers, just give them all a try.

(Pause)

OK, I guess we have to resume now, or we shall be too late for lunch. Now, we will let the seven exercises remain on the screen, while I give you some meta-exercises, exercises **about** these exercises. First:

Meta-exercise 0: Why did I give you these silly exercises?

Think a little about this. After the lecture, you may think again, and see if you have changed your mind.

(Brief pause)

Two more meta-exercises to go:

Meta-exercise 1: Which of these exercises were mathematical?

And

Meta-exercise 2: Which exercises had anything at all to do with mathematics?

Again, I will give you a couple of minutes to review the exercises in the light of the meta-exercises. Do feel free to discuss with your neighbours.

(Pause)

Time to move on again. Now we are getting really advanced: Here comes the

Meta-meta-exercise: How did you solve the meta-exercises? How do you distinguish between mathematical exercises, exercises with both mathematical and non-mathematical content, and exercises having nothing to do with mathematics?

Take a minute or two to think about this. And if you are getting nervous about the meta-meta-meta-exercise, the meta-meta-meta-meta-exercise etc., just take it easy. I promise you, we shall not climb any higher on the ladder of abstraction.

(Pause)

Done?

Of course not. I have thought consciously about this for about 35 years (and subconsciously since early childhood), and I still feel there is more to find out about it. So I certainly do not expect you to figure it all out in two minutes. Like all other endeavours of life, philosophy of mathematics is not so much about the final answers, whatever they may be, as the process of searching for temporary, partial answers, posing new and maybe more precise questions, and so on. Thus, whatever you thought about this last question, I can congratulate you on having done some philosophy of mathematics!

Indeed, the meta-meta-exercise poses one of the core questions of the philosophy of mathematics: What, if anything, distinguishes mathematical knowledge and activities from other kinds of knowledge and activities? Another question, which may be regarded as the reverse side of the same coin, is: What, if anything, is common to mathematical knowledge and activities, and this or that other kind of knowledge and activities? A third question, forming an almost Hegelian synthesis of the two first, is: What connections are there between mathematical knowledge and activities, and other kinds of knowledge and activities?

Before going on, I shall make brief comments on each of the seven exercises, not to give you "the one and only correct answer", but to satisfy some of your curiosity, give you some more food for thought, and perhaps clear up a few possible misunderstandings.

1. Here I am asking for your emotional reaction to a piece of music. Of course, there are deep connections between music and mathematics, but those connections would seem to have very little, if any, bearing on our emotional response, as a quality of a specific experience.

2. This involves the most elementary of all mathematical activities: counting. Thus, this exercise obviously has something to do with mathematics, and at kindergarten level it might even count as a mathematical exercise. But with a view to our search for distinguishing characteristics of mathematics, we should also ask for any peculiarity of this counting procedure. One very outstanding feature is its **formality**. When you tried to estimate the number of people in this room, did you take their respective age, size, gender or clothing into account? Of course not! A young beautiful girl counts just as much as an ugly old troll like me. You abstracted from almost all the information your senses were gushing upon you. The people were no longer individuals in all their glorious inequality, they were reduced to abstract units to be successively added to an equally abstract set. In short, you were focusing on one single, tiny, purely formal aspect of your experience, leaving out all other aspects as irrelevant to the task at hand.

3. Here we are comparing two different experiences as to their painfulness. Again we are disregarding the full content of the experiences, trying to put them on a common scale of pain. The comparison is qualitative, not quantitative, so we are not using the full concept of number. But we are in fact using a rather advanced concept, which in abstract algebra is called "partial order". This very general concept can also incorporate the possibility that toothache and headache are not comparable with regard to painfulness. Thus, improbable as it may seem, this exercise could in fact be interpreted as having something to do with mathematics, more precisely, abstract algebra. However, as it only involves a direct application of an idea capable of being moulded into an abstract mathematical concept, without any argument or reasoning occurring, regarding it as a mathematical exercise would be a little far-fetched. But if we had elaborated it a little, saying for instance: "If toothache is worse than stomach-ache, and stomach-ache is worse than headache, what is then worse of toothache and headache?", it might be construed as a simple mathematical exercise, its answer depending on whether the relation "worse than" is assumed to have the transitivity property we normally take for granted in any (potentially) numerical scale. If a mouse is smaller than a

cat, and a cat is smaller than an elephant, then, lo and behold, a mouse is smaller than an elephant! What a mind-shattering insight!

4. This is a simple counting exercise, but it has a curious quality of "self-reference", it is a sentence asking about itself. Such self-reference is far from unknown in our daily thinking, we are (or at least we should be) constantly questioning our own words as we are saying them. But here the self-reference is very explicit. Also, it is somewhat paradoxical in that the answer may vary with the language used, and even with different ways of putting it in the same language. The answer to "How many words are there in this sentence?" is obviously 8, but if we had said essentially the same thing this way: "How many words does this sentence contain?", the answer is suddenly 7. My French is rusty, to put it mildly, but I hope this is not too far from being correct: "Combien de mots y a-t-il dans cette phrase?" Here, if we count a-t-il as two words, the answer is 9. In my language, Norwegian, we do not have the "do-construction", so we could say, translated word by word: "How many words contains this sentence?", which brings the count down to 6.

Self-reference can also lead to real paradoxes, as in "This sentence is false". If it is true, then it is false, but if it is false, then it is true! Another version is the so-called "Liar's paradox". If I say: "I am lying", then if I am lying I am telling the truth, but if I am telling the truth, I am lying! What a mess!

5. This one was a real headache, perhaps worse than a toothache, for the ancient Greeks. On the one hand, they could easily construct a right-angled triangle where the two perpendicular sides were of equal length, say one meter, or whatever length unit they had. Of course, they knew their Pythagoras very well: the square on the hypotenuse is the sum of the squares on the two perpendicular sides. This sum is clearly 2. Thus, the length of the hypotenuse, measured in the same unit as the other sides, should be a number which multiplied by itself equals 2.

On the other hand, their number concept included the natural numbers, i.e. the numbers we use in counting, 1, 2, 3, etc., and fractions of these. Obviously, there is no natural number which multiplied by itself equals 2. But they could also prove that there is no fraction which multiplied by itself equals 2. They could get as close as they wanted, but they could never get exactly 2.

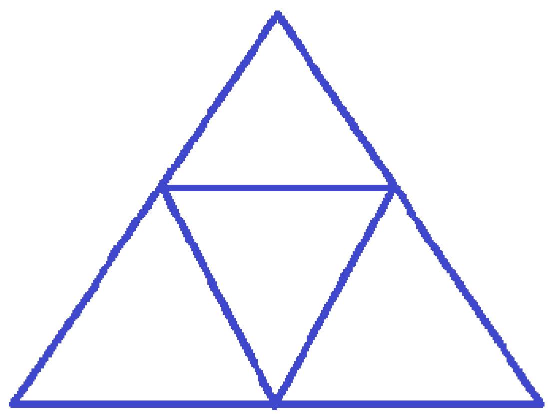
Euclid developed a complex theory of "proportions" to deal with this difficult situation. These "proportions" were in effect a new kind of "numbers", of which the natural numbers and the fractions formed only a small part. Within this new expanded set of "numbers" there is a number which multiplied by itself is exactly 2.

So, the answer to the exercise is no if you only accept natural numbers and fractions as "numbers", and yes if you also accept this new kind of numbers. Nowadays we have other, simpler ways of constructing the so-called "real numbers", and we have also made several further expansions of the number concept.

6. Here, I have been really naughty. I hope you did not waste too much time on this one. There simply are no plane biangles! Try as you might, you have to have at least three straight lines to get a closed figure. Why? Alas, I cannot give you a satisfactory answer to that simple question here. I can only say that it has to do with some deep properties of our immediate intuition of space.

Some of you may have heard of "non-Euclidean geometries". The geometry we learned at school was the good old Euclidean one, and until about two centuries ago there was no other geometry. But problems related to Euclid's parallel postulate eventually led some brave souls to entertain the idea of alternative geometries, which quite naturally were dubbed "non-Euclidean". Now, in one of these alternative geometries plane biangles are in fact possible! As it happens, this geometry can be fairly accurately depicted in Euclidean spherical geometry. On a sphere, of course, there are no straight lines. But if you cut a sphere with a plane, the cutting curve will be a circle, and if the plane goes through the centre of the sphere, the circle will have the same radius as the sphere itself. Such circles are understandably called "great circles" on the sphere. If we ignore the fact that the earth is not a perfect sphere - like some of its inhabitants it is a bit fatter round the middle - we can think of the equator and the meridians. Now, if we accept the great circles as "best substitutes" for straight lines on the sphere, we see that any two such "straight line substitutes" will indeed give us a "spherical biangle". For instance, the meridian through Greenwich and the meridian through St. Petersburg give a biangle containing Barcelona and the easternmost part of Spain, most of France, a little bit of England, and almost all the rest of Europe, except Portugal. It would also include a large chunk of Africa, with Libya, Tunisia, Chad, Nigeria, etc.

7. This one is simple, if you draw the right lines. I guess the figure says it all:



Each of the small triangles is equilateral, like the large one - well, at least if you disregard my ineptitude at drawing. What is really important here is to realize that we could repeat the process ad indefinitum, getting smaller and smaller triangles, and we could also go the other way, regarding the large triangle as part of an even greater one, and repeat that process too, getting larger and larger triangles. This reveals a very important quality of space, which I have called "homogeneity of scale": no matter how much we zoom in or out, space always "look the same". This is not the case for the non-Euclidean geometries. Thus, in my opinion, only Euclidean geometry describes the space of our intuition, space as immediately perceived by us.

Now, back to the peculiarities of mathematical knowledge. Consider a very simple example: At school I learned that there are nine planets in our solar system. Since then, poor Pluto has been robbed of its planethood, so now there are eight. I don't know about you, but I feel that the number of planets in the solar system is a rather arbitrary fact (or "contingent", as the philosophers like to say). There might have been five, or seven, or fifteen, or whatever. Probably, there are other solar systems out there with any of these numbers of planets. Now, I also learned that 2 plus 2 equals 4. Could this also have been different? Could there be some other place in the universe, or some other conceivable universe, where 2 plus 2 equals 3, or 5? I think most of us would say no. But if asked why, most of us would be hard put to come up with a cogent answer. Besides, let us apply this simple arithmetical fact, for instance to "live animals": 2 live animals plus 2 live animals equals 4 live animals. In most cases, this is undeniably true. But what if the two first animals were hungry wolves and the other two were sheep? Well, in the garden of Eden this might still give four live animals, but in our brutal world the most probable result of this calculation is two happy, satisfied wolves and two eaten-out sheep carcasses.

I think most of you would agree with me that this case does not constitute a counterexample to the general truth of 2 plus 2 equals 4. Why? I cannot give you my complete answer here, just hint that, in my opinion, arithmetic is about successions, not about the fortunate relative stability of the objects in our world. And after all, the wolves and the sheep would remain four live animals – two salivating wolves and two terror stricken sheep - for a few seconds.

If "2 plus 2 equals 4", and maybe all other mathematical truths as well, really could not have been otherwise, these truths clearly differ from most other truths we know. While a physical, psychological, astronomical, or historical assertion may be true, a mathematical assertion would seem to be not only true, but **necessarily true**. In philosophy, such assertions are called **a priori**. Mind you, that an assertion is a priori only means that if it is true, it is necessarily true. It does not mean that it is obviously true, or that we can be absolutely sure that it is true. This is, in fact, a common misunderstanding, even among distinguished professors of philosophy! So, now you can brag about knowing more than they do! But don't tell them I told you, or they will pursue me for the rest of my life...

So, our gut feeling, as amateur mathematicians, is that mathematical statements are a priori in this specific sense, and I can tell you, without a shadow of a doubt, that the more mathematical experience you get, the more certain you will be about this gut feeling. Nevertheless, contrary to the situation just two centuries ago, most philosophers of mathematics today reject it. In fact, I find myself belonging to the endangered species of "a priorist" philosophers. Not only do I assert that mathematics is a priori, I also claim to have incontrovertible proof of this alleged fact, thereby causing such vigorous headshaking by the majority of my colleagues that I am afraid they may injure their necks.

I cannot go further into this issue here. Instead, I shall briefly sketch the main historically occurring positions in this field, with the question of whether mathematics is a priori as a convenient starting point.

Ok, we have to decide what to do about this gut feeling. If we choose to accept it, then it is natural to ask whether there are other a priori sciences. The most obvious candidate is, of course, logic. I mean, if a bachelor is defined as an "unmarried man", then the claim "all bachelors are unmarried" may not be very enlightening, but it is most certainly true. Moreover, it is **necessarily true**, it is absolutely inconceivable that we would ever find a counterexample, i.e. a married bachelor. Thus, logic would seem to be even more obviously a priori than mathematics.

With two a priori sciences, what is the relationship between them? One possible answer is that mathematics is deducible from, or even essentially identical with, logic. This is the position called **logicism**. The German philosopher Gottlob Frege tried to carry out the deduction of a part of mathematics, arithmetic, from logic. His attempts ended up in self-contradictions, the so-called paradoxes of set theory. So, logicism did not seem to work very well. Nonetheless, a reduced version of logicism, "**if-then-ism**", is more or less enthusiastically endorsed by most mathematicians. This approach admits the possibly non-logical character of mathematical **axioms** and **theorems**. Only mathematical **proofs** or **inferences**, i.e. "if-then-statements", are still said to be derivable from pure logic. Everybody agrees that the old Aristotelian logic does not suffice, but most philosophers and mathematicians say that with modern so-called second-order quantificational logic, the task is still formidable, but feasible. I beg to disagree. My contention is that many inference patterns now counted as "logical", are not purely logical at all, but distinctly mathematical, so that we really end up saying that mathematics can be derived from mathematics, which does not seem very surprising.

Thus, despite its popularity among mathematicians, if-then-ism does not seem to offer a viable way of validating our gut feeling about mathematical statements being a priori, i.e. necessarily true, if true. But what about going all the way? We have already noticed the curious **formal** character of mathematics. Now, instead of just retreating from asserting the truth of the axioms and the theorems as independent statements, but still assert the allegedly logical connections between them, we could retreat from asserting anything at all. To see that this seemingly absurd position is even conceivable, we have to consider a simple but easily overlooked fact: To be expressible and communicable at all, any mathematical definition or statement has to be expressible as a finite sequence of symbols. This is not very surprising. After all, even a sonnet by Shakespeare is expressed as a finite sequence of letters, spaces, and punctuation marks.

So, what we need is a finite set of symbols, rules for combining those symbols into terms, rules for combining terms into definitions and statements, some finite set of statements called axioms, and finally rules for "valid" transitions from a statement or set of statement to a new statement. I am omitting a lot of intricate details here, but this is enough to get the general picture needed for our introductory purposes. Such a collection of symbols and rules we may call a formal system, and the program of **formalism**, a way of thinking in the philosophy of mathematics instigated by the great mathematician David Hilbert, is to translate all of current (and future) mathematics into such formal systems.

If Hilbert's vision had come to fruition, everything would have been rather transparent. We would simply resort to juggling finite sequences of symbols, meticulously observing the rules of the game. And that is all that is left: a game just like chess, a lot more complex and intricate, but yet a game. Hmm, this seems to fit rather well with the picture most people have of what mathematicians are doing…

But formalization proved to be more powerful than Hilbert had bargained for. Indeed, it was powerful enough to prove its own limitations! In the 1930's, a young German mathematician, Kurt Gödel, was able to construct a formal statement essentially saying of itself: "I am not provable". He was also able to prove that if a formal system strong enough to be of any mathematical interest is consistent, i.e. free from contradictions, then this can never be proved within the formal system itself. Thus, formalization can never be complete, and formalism, though elucidating the essential feature of formality, did not give us a completely satisfying picture of the nature of mathematics.

Now we seem to be back at square one. Do we really have to reject our gut feeling that mathematics is a priori? In fact, a majority of contemporary philosophers of mathematics draw that conclusion, and most of the current discussion in the field is really between different versions of **empiricism**. The central thesis of empiricism is that mathematical statements, like 2+2 = 4, are empirical, on a par with physical laws like the law of gravity or astronomical facts like the number of planets in our solar system. Well, how do empiricists account for our persistent gut feeling? Of course, there are variations, but the common denominator of their answers is that some portions of our experience are more certain and therefore more immune to change, than others. Our total body of knowledge could be compared to an onion, where the outer layers are most liable to be changed by new experience. Logic and mathematics belong to the innermost core, and are therefore far less likely to undergo any substantial corrections, no matter how bizarre our future experience may be. But even here no absolute guarantee against future corrections can be issued. Our gut feeling is therefore correct in giving us greater faith in mathematical facts than in most other facts we encounter. It is just not correct in asserting the **absolute necessity** of these facts.

Nice try, but no thanks! In most cases, I am willing to negotiate to reach an agreed compromise, but when it comes to truth, I am a hard nail. And I just don't buy this. To me, the difference between logic and mathematics on the one hand, and other fields of knowledge on the other, is a difference of species, not one of degree. Therefore, I am obliged to reject empiricism out of hand.

Fortunately, we have not yet exhausted all possibilities. Hitherto, we have all but ignored the role of the human mind. We have talked about logic, formal systems, and experience, as some abstract entities existing "out there". But of course, it all goes on in our minds. A textbook of mathematics is no different from a novel or a dictionary until someone reads it and hopefully understands it. Thus, mathematical objects and processes would seem to be a kind of **mental** entities and activities. This is the view of **intuitionism**, instigated by the Dutch mathematician Brouwer.

For intuitionism, mathematical objects and operations are a certain kind of **mental constructions**. "Construction" and "constructibility" are the core concepts. The plane biangle does not exist because it cannot be constructed. On the other hand, we can easily construct a right-angled triangle with the two perpendicular sides of equal length, and the length of the hypotenuse gives a number which multiplied by itself equals 2, so this number does exist.

This scheme seems to work very well. But there are problems with this demand for constructibility. For instance, the life-time of the human race is probably limited. Therefore, most natural numbers will never be explicitly constructed by any human being. But don't they exist?

As you may have guessed, I am not altogether satisfied with the intuitionist approach either. And now I am running out of time, so this will be the last approach I will tell you about. But before I end the lecture, I feel obliged to at least hint at my own preferred approach.

Brouwer's intuitionism is quite explicitly inspired by Immanuel Kant's account of mathematical knowledge. In Kant too, the concept of construction is crucial. Also, he would partly agree with Brouwer as to the "non-logical", yet a priori, character of mathematics. However, in my opinion Brouwer both oversimplifies and complicates the Kantian concept of construction, thereby making his approach very vulnerable to empiricist assaults. I mean, we may grant that our mental operations are a part of nature, but why should other parts of nature, as apples or stones, conform to our mental predilections? Is it not more probable that these predilections have developed in response to our prolonged experience with apples, stones, and other natural phenomena?

To counter such objections, we have to dig deeper into the conditions of the possibility of experience itself, and that is exactly what Kant does. His philosophy of mathematics flows naturally from these deep investigations, and in my opinion gives a far more adequate account of mathematical knowledge than any of the other approaches, which is quite remarkable, given the enormous development in mathematics after his time (he died in 1804). But I can say no more here. If you are interested, shoot me an e-mail, and I will happily deluge you with ample material on this issue.

Thank you for your attention!